Time correlations of a laser–induced Bose–Einstein condensate

Timo Felbinger†, Luis Santos‡, Martin Wilkens† and Maciej Lewenstein‡
†*Universität Potsdam, Fachgruppe Physik, Am Neuen Palais 10, D-14469 Potsdam, Germany*
‡*Institut für Theoretische Physik, Universität Hannover, Appelstr. 2, D–30167 Hannover, Germany*

We analyze the multi–time correlations of a laser–induced Bose–Einstein condensate. We use quantum stochastic methods to obtain under certain circumstances a Fokker–Planck equation which describes the phase–difussion process, and obtain an analytical expression of the two–time correlations. We perform also quantum Monte Carlo numerical simulations of the correlations, which are in good agreement with the predicted analytical results.

I. INTRODUCTION

During the last years the combination of laser cooling [1] and evaporative cooling methods [2] has lead to the achievement of one of the most pursued goals since the early days of quantum physics, i.e. the so–called Bose–Einstein Condensation (BEC). Such condensation is a direct consequence of the Bose–Einstein statistics, and consists in the fact that the ground state of the system becomes macroscopically populated. Although BEC has been obtained using the combination of laser and collisional techniques, there are several experimental groups [3–6,8] which currently investigate the possibility of obtaining BEC using all–optical means only. In such a case the number of atoms in the trap would not decrease during the cooling process, and a non–destructive detection of BEC could be performed by simply measuring the fluorescence photons. In addition, laser induced condensation would be easier to control externally, and could lead to richer effects than collisional processes, as those employed in evaporative and sympathetic cooling. Finally, the methods employed to achieve laser induced condensation can be used to design the techniques of pumping the atoms into the condensate using spontaneous emission [7,9–11].

The most sophisticated laser cooling techniques (as Velocity Selective Coherent Population Trapping [12] or Raman cooling [13]) are capable to cool atomic samples below the photon–recoil energy, $E_R = \hbar\omega_R = \hbar^2 k_L^2/2m$, where k_L is the laser wavevector, and m is the atomic mass. Such methods should in principle lead to BEC. However, in using laser cooling to obtain the BEC, the reabsorption of spontaneously emitted photons turns to be a very important problem. This is because the subrecoil laser cooling techniques are based on dark–state mechanisms [14], and the dark–states are unfortunately not dark respect to the spontaneously emitted photons. Therefore, multiple reabsorptions can increase the system energy by several recoil energies per atom [15–18]. It is easy to understand that assuming that the reabsorption cross section for trapped atoms is the same as in free space, i.e. $\simeq 1/k_L^2$, the significance of reabsorptions increases with the dimensionality [5,16], in such a way that the reabsorptions should not cause any problem in one dimension. For the case of three–dimensional traps several remedies have also been proposed [19]. In the following we are going to assume that our system is one–dimensional. This assumption in the first place has been done for simplicity reasons, but at the same time it allows us to avoid the reabsorbtions problem. As we discuss in the conclusion section, however, our approach can be used in three–dimensional asymmetric traps, for which heating due to reabsorbtions can also be negligible.

When laser–cooling methods are employed, the thermalization of the system occurs with the interaction with the laser field and the vacuum modes of the electromagnetic field, and therefore atom–atom collisions are in principle not necessary. In the following we shall consider an ideal gas (see the discussion in Sec. II). In fact, the proper notion of temperature must then be revised. On the other hand, the dynamics of the system becomes more complex, because we have to deal with an open system which interacts with the laser. In Refs. [20,21], the quantum dynamics of a laser–cooled ideal gas has been studied using second quantization formalism in order to take into account the quantum–statistical character of the bosons, and also employing quantum stochastical methods [22–24] to take into account the coupling with the reservoir provided by the vacuum modes of the electromagnetic field. The equations which describe the dynamics of the system have been obtained, and so has the stationary solution, which corresponds for the case of a bosonic system with the well–known Bose–Einstein distribution (BED), and under proper conditions the BEC appears.

In order to characterize this laser–induced BEC, the two–time correlations of the condensate amplitude turn to be very important quantities, because they describe the diffusion of the phase of the condensate. These correlations are very difficult to calculate both analytically and numerically, and until now no calculation of such correlations has been published, as far as we know. The main aim of the present paper is to present a method to calculate such correlations using quasi–probability representations, and in particular P–representation [23]. We shall also present a numerical calculation based on the so–called wave–function Monte Carlo method [25], whose

results are in good agreement with the predicted analytical ones.

At this point it is worth noticing that the model considered in this paper is not very realistic since it is one dimensional, and even in the relevant single dimension it assumes the, so called, Lamb-Dicke limit, when the size of the trap a is smaller than the wavelength of the laser field, $\lambda/2\pi$. At the same time the model neglects not only elastic collisions, but also all non-elastic losses: two– and three–body non-elastic collisions, as well as photoassociation losses. Realization of such situation requires modification of atomic scattering length (as discussed in Section II and conclusions)and other precautions, which are possible (as discussed in Section II and in more detail in conclusions), but not easy to achieve experimentally. In the worst case, however, the model is realistic for small samples of $N = 15 - 50$ atoms. Cooling of such small sample of atoms to a ground state of the trap is of fundamental interest itself, and might be even useful for applications for quantum information processing [26]. We stress, however, that the main achievement of this paper concerns *methodological aspects*. Even though the model is somewhat unrealistic, it is a paradigm model describing quantum dynamics of the bosonic gas approaching the equilibrium. As we mentioned, time correlations are very difficult to calculate analytically and numerically in any model of that sort, regardless how realistic it is. We develop here the methods to calculate such correlations, and these methods are of quite importance themselves, since they are quite universal and can be carried over to more realistic models, such as those discussed by us in a series of Refs. [27].

The structure of the paper is as follows. In Sec. II, we briefly review the calculations already presented in Ref. [20]. The P–representation formalism is introduced in Sec. III. The expressions obtained in Sec. III, allows us in Sec. IV to calculate the two–time correlations of the amplitude of the condensate. In Sec. V, we present the employed numerical method, and the comparison between numerical and analytical results. We conclude in Sec. VI stressing once more methodological achievements of this paper, and discussing experimental feasibility of our model.

II. MODEL

In this section we briefly review the formalism already developed in Ref. [20]. We consider a system composed of N two–level identical bosons in an harmonic trap centered at $\vec{R} = 0$, which are coherently driven by a standing wave laser field, and also interact with the vacuum modes of the electromagnetic field. In the following we use, for simplicity, units with $\hbar = 1$, and velocity of light $c = 1$. As in Ref. [20], we consider four basic approximations: (i) the laser is quasi–resonant with a particular transition between two electronic states $|g\rangle$ (ground state) and $|e\rangle$ (excited state), and therefore we consider atoms as two–level atoms; (ii) Rotating–wave approximation (RWA) can be used, since the frequency of the considered electronic transition (ω_0) and the laser frequency ω_L are considered much larger than any other frequency scale of the system; (iii) we treat the atom–field interactions in the dipole approximation since the resonant wavelength $\lambda_L \gg a_0$, where a_0 is the typical atomic size; (iv) we neglect the atom–atom collisions, i.e. we work in the ideal–gas approximation. In principle, the latter approximation implies that we should deal with a small number of particles ($N < 100$), because we are going to consider that the trap is in the so–called Lamb–Dicke Limit (LDL), where the, so called, Lamb–Dicke parameter $\eta = 2\pi a/\lambda_L < 1$, with a being the size of the trap ground state. However, the s–wave scattering length a_{sc} (which governs the atom–atom interactions at low energies) can be externally modified, for instance, using a magnetic field [28,29], or a laser field [30], and in principle a_{sc} can be made very close to zero [31,32], allowing the strict validity of the ideal gas approximation.

We limit our discussion to the one–dimensional case. The relevant electronic states are then determined by the laser polarization perpendicular to the laser wave vector, which in turn can be aligned in the z–direction, $\vec{k}_L = (0, 0, \omega_L)$. We assume that the atoms can only move in the z–direction, and are localized in transverse direction, so that practically one can assume that they are located at $\vec{R} = (0, 0, z)$. The atoms are assumed to occupy the energy eigenstates of the harmonic trap potential. We denote this energy levels by $|l, g\rangle$ ($|m, e\rangle$) for the atoms in the ground (excited) electronic state, occupying the level $l(m)= 0, 1, \ldots$ of the harmonic trap.

We introduce also the operators g_l, $g_l^\dagger$ (e_m, $e_m^\dagger$) that annihilate or create atoms in the lth (mth) energy level of the ground–state (excited–state) potential. These operators fulfill standard commutation relations for bosonic atoms:

$$[g_l, g_{l'}^\dagger] = \delta_{ll'}, \tag{1}$$

$$[e_m, e_{m'}^\dagger] = \delta_{mm'}. \tag{2}$$

Using standard quantum stochastic methods [22–24] one can eliminate the vacuum modes of the electromagnetic field, and after performing a systematic expansion in the Lamb–Dicke parameter η one obtains [20] the master equation (ME), in the frame rotating with the laser, valid up to the order $\mathcal{O}(\eta^2)$:

$$\dot{\rho} = -i[\mathcal{H}_a + \mathcal{H}_{las}, \rho] + \mathcal{L}\rho, \tag{3}$$

where the atomic part of the Hamiltonian takes the form

$$\mathcal{H}_a = \sum_{m=0}^{\infty} \omega_m g_m^\dagger g_m + \sum_{m=0}^{\infty} (\omega_m - \delta) e_m^\dagger e_m, \tag{4}$$

with $\delta = \omega_L - \omega_0$ being the laser detuning. In Eq. (4), the center–of–mass potentials for ground and excited atoms

can be well approximated by harmonic potentials of frequency $\omega_g = \omega_e = \omega$ plus a small anharmonicity so that the energy levels are $\omega m + \omega\alpha m^2$ with $m = 0, 1, \ldots$ and $\alpha \ll 1$. From now on we use the same indices for the energy levels in the ground– and excited potentials.

The Hamiltonian of the atom–laser interaction takes the form:

$$\mathcal{H}_{las} = \eta\frac{\Omega}{2}\sum_{m}^{\infty}\sqrt{m+1}\left(e^{\dagger}_{m+1}g_m + e^{\dagger}_m g_{m+1} + H.c.\right) + \mathcal{O}(\eta^3), \quad (5)$$

where Ω denotes the Rabi frequency describing the laser–atom interaction.

The spontaneous emission part in Eq. (3) takes the form:

$$\mathcal{L}\rho = \frac{\Gamma}{2}\sum_{m,m'}\Big(2g^{\dagger}_m e_m \rho e^{\dagger}_{m'} g_{m'} - e^{\dagger}_{m'} g_{m'} g^{\dagger}_m e_m \rho - \rho e^{\dagger}_{m'} g_{m'} g^{\dagger}_m e_m\Big) + \mathcal{O}(\eta^2), \quad (6)$$

where Γ is the natural line-width of the considered transition.

If we locate the trap at the node of the laser standing wave, the excitation provided by the laser then become weak, provided that Ω is moderate. We expect then that at a given instant no more than one atom will be excited. The system is then characterized by two distinct time scales: the fast one, determined by $N\Gamma$, ω and δ; and the slow one, characterized by $\eta^2\Omega^2/N\Gamma, \eta^2\Omega^2/(\omega \pm \delta)$, which describes the jumps between the various ground–state levels. In such a case one can use standard adiabatic elimination techniques [22] to remove excited state populations. Physically, this reflects the fact that a combined process of excitation and (relatively rapid) spontaneous decay causes a redistribution of atomic population among different levels of the ground–state trap. The ME after the adiabatic elimination of the excited states becomes:

$$\dot{\rho} = \sum_{m}^{\infty}\mathcal{L}_m\rho, \quad (7)$$

where the coupling between the levels m and $m+1$ [33] is given by [20]:

$$\mathcal{L}_m\rho = \frac{\Gamma_-}{2}(m+1)(2A_m\rho A^{\dagger}_m - A^{\dagger}_m A_m\rho - \rho A^{\dagger}_m A_m) + \frac{\Gamma_+}{2}(m+1)(2A^{\dagger}_m\rho A_m - A_m A^{\dagger}_m\rho - \rho A_m A^{\dagger}_m), \quad (8)$$

with $A_m = g^{\dagger}_m g_{m+1}$, and

$$\Gamma_{\pm} = \Gamma(\eta\Omega/2)^2\frac{1}{(N\Gamma/2)^2 + (\omega \mp \delta)^2}. \quad (9)$$

It is easy to check [20] that for $\delta < 0$ there exists an exact steady state solution of ME (7) which has the canonical form:

$$\rho_{st} = \frac{1}{Z}q^{\sum_m m g^{\dagger}_m g_m}, \quad (10)$$

where $q = \exp(-\omega/k_BT) = \Gamma_+/\Gamma-$, and Z is the canonical partition function. For the particular case of a one–dimensional harmonic potential the partition function can be derived in closed form [35]:

$$Z = \prod_{j=1}^{N}\frac{1}{1-q^j}. \quad (11)$$

The thermodynamics of the system is determined by the Helmholtz free energy $F = -k_BT\ln(Z)$. The canonical version of the chemical potential is then given by:

$$\mu \equiv (\partial F/\partial N)_{T,\omega} = k_BT\ln[1-q^N]. \quad (12)$$

There is a non–zero temperature [36]

$$T_c = \frac{\hbar\omega}{k_B}\frac{N}{\ln(N)}, \quad (13)$$

below which μ is effectively zero. q can be conveniently re–written in terms of this critical temperature:

$$q = \exp\left(-\frac{\ln(N)}{N}\frac{T_c}{T}\right). \quad (14)$$

III. P–REPRESENTATION

In the following section, we are going to employ ME (7) to calculate the two–time correlations of the ground state in the stationary regime, i.e. correlations of the form $\langle g^{\dagger}_0(t)g_0(0)\rangle$. In order to do that it is convenient to use the so–called Glauber–Sudarshan P–representation [23], which is defined as:

$$\rho(t) = \int d^2z_0 d^2z_1 \ldots P(z_0, z_0^*; z_1, z_1^*; \ldots) \times |z_0, z_1, \ldots\rangle\langle z_0, z_1, \ldots|. \quad (15)$$

where $|z_0, z_1, \ldots\rangle$ are coherent states. After transforming into P–representation, Eq. (7) becomes:

$$\dot{P} = \sum_{m=0}^{\infty}\hat{S}_m P, \quad (16)$$

where the transition between m and $m+1$ is given by the operator:

$$\hat{S}_m = \frac{\Gamma_-}{2}(m+1) \Bigg\{2\left[|z_m|^2 - \left(\frac{\partial}{\partial z_m}z_m + \frac{\partial}{\partial z_m^*}z_m^*\right) + 1 + \frac{\partial^2}{\partial z_m \partial z_m^*}\right]|z_{m+1}|^2 - \left(|z_{m+1}|^2 - \frac{\partial}{\partial z_{m+1}}z_{m+1}\right)\left(|z_m|^2 - \frac{\partial}{\partial z_m}z_m + 1\right)$$

$$
- \left(|z_{m+1}|^2 - \frac{\partial}{\partial z^*_{m+1}} z^*_{m+1}\right)\left(|z_m|^2 - \frac{\partial}{\partial z^*_m} z^*_m + 1\right)
$$
$$
2q\left[|z_{m+1}|^2 - \left(\frac{\partial}{\partial z_{m+1}} z_{m+1} + \frac{\partial}{\partial z^*_{m+1}} z^*_{m+1}\right) + 1 + \frac{\partial^2}{\partial z_{m+1}\partial z^*_{m+1}}\right]|z_m|^2
$$
$$
- \left(|z_m|^2 - \frac{\partial}{\partial z_m} z_m\right)\left(|z_{m+1}|^2 - \frac{\partial}{\partial z_{m+1}} z_{m+1} + 1\right)
$$
$$
- \left(|z_m|^2 - \frac{\partial}{\partial z^*_m} z^*_m\right)\left(|z_{m+1}|^2 - \frac{\partial}{\partial z^*_{m+1}} z^*_{m+1} + 1\right)\Bigg\} \quad (17)
$$

Let us define the reduced P–representation for the ground state of the trap as:

$$
P_0(z_0, z_0^*; t) = \int d^2 z_1 d^2 z_2 \ldots P(z0, z0^*; z1, z1^*; \ldots; t). \quad (18)
$$

The equation which describes the dynamics of P_0, takes the form:

$$
\begin{aligned}
\dot{P}_0 = {} & \frac{\Gamma_-}{2} q \left(\frac{\partial}{\partial x_0} x_0 + \frac{\partial}{\partial x_0^*} x_0^*\right) P_0 \\
& + \frac{\Gamma_-}{2}(q-1)\left(\frac{\partial}{\partial x_0} x_0 + \frac{\partial}{\partial x_0^*} x_0^*\right) P_1 \\
& + \frac{\Gamma_-}{N} \frac{\partial^2}{\partial x_0 \partial x_0^*} P_1. \quad (19)
\end{aligned}
$$

where $P_1 = \int d^2 z_1 d^2 z_2 \ldots |z_1|^2 P$. We have also introduced here the convenient notation $z_0 = \sqrt{N} x_0$.

One can prove (see Appendix A) that for times $t > [\Gamma_-(1-q)N|x_0|^2]^{-1}$, one can adiabatically eliminate the excited trap states, and retrieve a closed Fokker–Planck equation (FPE) for the reduced P_0:

$$
\dot{P}_0 = \frac{\Gamma_- q}{N(1-q)} \left\{ \frac{\partial^2}{\partial x_0 \partial x_0^*} - \frac{1}{4}\left[\frac{\partial}{\partial x_0} x_0 + \frac{\partial}{\partial x_0^*} x_0^*\right]\left[\frac{\partial}{\partial x_0} x_0 + \frac{\partial}{\partial x_0^*} x_0^*\right] \frac{1}{|x_0|^2} \right\} P_0. \quad (20)
$$

Changing into the more convenient variables $z_0 = r\exp(i\phi)$, the FPE (20) becomes:

$$
\dot{P}_0 = \frac{\mathcal{K}}{r^2} \frac{\partial^2}{\partial \phi^2} P_0, \quad (21)
$$

with

$$
\mathcal{K} = \frac{\Gamma_- q}{4(1-q)}. \quad (22)
$$

Eq. (21) is one of the central results of this paper, and has a simple physical interpretation as an equation describing diffusion of the phase of the condensate wave function. Interestingly, even though we are dealing here with an ideal gas approaching the thermal equilibrium, the equilibrium state has temporal properties characteristic for the states obtained via spontaneous breaking of the $U(1)$ phase symmetry, such as in the theory of laser, or in general any theory of second order phase transitions in which the effective potential for the order parameter below the transition temperature has a Mexican hat shape [34]. In particular, the phase diffusion rate is here inverse proportional to r^2, i.e. to the number of atoms in the condensate.

IV. TWO–TIME CORRELATIONS OF THE CONDENSATE

The solution of the FPE (21) can be obtained using the following Green function:

$$
\begin{aligned}
& G(r, \phi, t || r_o, \phi_0, t = 0) = \\
& \frac{1}{\pi r_0} \delta(r - r_0)[\frac{1}{2} + \sum_{n=1} \cos n(\phi - \phi_0) e^{-n^2 \frac{\mathcal{K}}{r_0^2} t}]. \quad (23)
\end{aligned}
$$

Using this we can already calculate the time correlation:

$$
\begin{aligned}
& \langle g_0^{\dagger l}(t) g_0^l(0)\rangle \\
& = \int_0^\infty \int_0^{2\pi} r dr d\phi \int_0^\infty \int_0^{2\pi} r_0 dr_0 d\phi_0 \\
& \times G(r, \phi, t || r_o, \phi_0, t = 0) P_0(r_0, \phi_0) r^l r_0^l e^{-il(\phi - \phi_0)} \\
& = \int_0^\infty \int_0^{2\pi} r dr d\phi r^{2l} e^{-l^2 \frac{\mathcal{K}}{r^2} t} P_0(r, \phi) \\
& = \langle : (g_0^\dagger g_0)^l e^{-l^2 \mathcal{K} t / g_0^\dagger g_0} : \rangle, \quad (24)
\end{aligned}
$$

where ":" denotes normal ordering. In the following we just consider the case of an exponent $l = 1$. We expand the exponent, assuming that for temperatures T sufficiently below T_c, fluctuations of $n_0 = g_0^\dagger g_0$ are small:

$$
\begin{aligned}
\frac{\mathcal{K} t}{n_0} & = \frac{\mathcal{K} t}{\langle n_0\rangle - (\langle n_0\rangle - n_0)} \\
& \approx \frac{\mathcal{K} t}{\langle n_0\rangle}\left(1 + \left(1 - \frac{n_0}{\langle n_0\rangle}\right)\right) \\
& = \frac{2\mathcal{K} t}{\langle n_0\rangle} - \frac{\mathcal{K} t}{\langle n_0\rangle^2} n_0 \,. \quad (25)
\end{aligned}
$$

Using this expression and the identity $: \exp(\xi g_0^\dagger g_0) : = \exp(\ln(\xi+1) g_0^\dagger g_0)$, Eq. (24) can be rewritten in the form:

$$
\begin{aligned}
& \langle g_0^\dagger(t) g_0(0)\rangle \\
& = \langle : g_0^\dagger g_0 \exp\left[-\frac{2\mathcal{K} t}{\langle n_0\rangle} + \frac{\mathcal{K} t}{\langle n_0\rangle^2} g_0^\dagger g_0\right] : \rangle \\
& = e^{-2\mathcal{K} t/\langle n_0\rangle} \langle n_0 \exp\left[(n_0 - 1)\ln(1 + \mathcal{K} t/\langle n_0\rangle^2)\right]\rangle \,. \quad (26)
\end{aligned}
$$

Note that from Eq. (26), it is clear that the correlations depend on the fluctuations of the condensate fraction, which are not correctly described using the usual text–book treatment of the system based on a grand–canonical

ensemble [35–39]. In order to calculate the correlations, it is thus necessary to calculate the required averages using physically sound ensemble, which in this case is the canonical one. Using canonical ensemble we have to determine the probabilities to have n_0 particles in the ground state, given a total number N of particles [36]:

$$P_0^{CN}(n_0|N) = q^{N-n_0} \prod_{j=N-n_0+1}^{N} (1-q^j). \qquad (27)$$

Therefore:

$$\langle (g_0^\dagger(t))(g_0(0)) \rangle = e^{-2\mathcal{K}t/\langle n_0 \rangle} \sum_{n_0=0}^{N} P_0^{CN}(n_0|N) n_0 \left(1 + \frac{\mathcal{K}t}{\langle n_0 \rangle^2}\right)^{n_0-1}. \qquad (28)$$

The above closed analytic formula is another central result of this paper.

V. NUMERICAL RESULTS

Numerical simulations of the system under consideration can be performed using Quantum Monte Carlo techniques. In our case, it is a quite difficult task due to the large size of the Hilbert space which has to be simulated, even if only a relatively small total particle number N and a low cutoff of the trap level structure is chosen.

To avoid having to compute the dynamics of the huge density matrix of this system, we applied a stochastic wave function method, which replaces the deterministic evolution of the density matrix (following Eq.(7)) by a stochastic evolution of an ensemble of state vectors [40].

A naive application of this algorithm to simulate (7) would lead to a stochastic jump process in which the operators A_m and $A_m^\dagger$ act as jump operators. This process would represent the density matrix as an ensemble of simultaneous eigenvectors of all occupation number operators $n_m = g_m^\dagger g_m$, causing large fluctuations of the phase which we are interested in. Consequently, the Monte Carlo simulations would require a very large number of realizations.

We found that convergence can be improved by rewriting the Liouvillian operators given in Eq. (8) in the following way:

$$\begin{aligned}
\mathcal{L}_m\rho &= \frac{\Gamma_-}{2}(m+1)(2A_m^{(+)}\rho A_m^{(+)\dagger} - A_m^{(+)\dagger}A_m^{(+)}\rho - \rho A_m^{(+)\dagger}A_m^{(+)}) \\
&+\frac{\Gamma_-}{2}(m+1)(2A_m^{(-)}\rho A_m^{(-)\dagger} - A_m^{(-)\dagger}A_m^{(-)}\rho - \rho A_m^{(-)\dagger}A_m^{(-)}) \\
&+\frac{\Gamma_+}{2}(m+1)(2A_m^{(+)\dagger}\rho A_m^{(+)} - A_m^{(+)}A_m^{(+)\dagger}\rho - \rho A_m^{(+)}A_m^{(+)\dagger}) \\
&+\frac{\Gamma_+}{2}(m+1)(2A_m^{(-)\dagger}\rho A_m^{(-)} - A_m^{(-)}A_m^{(-)\dagger}\rho - \rho A_m^{(-)}A_m^{(-)\dagger}), \qquad (29)
\end{aligned}$$

with operators $A_m^{(\pm)} := \frac{1}{\sqrt{2}}(A_m \pm \eta_m)$, where the η_m are c-numbers which are chosen to be of the same order of magnitude as the mean occupation number $\sqrt{\langle n_m n_{m+1}\rangle}$, so that $\eta_m \approx A_m$. Eq. (29) has an advantage that while it is, of course, reproducing the same master equation Eq. (7), it leads to a different stochastic jump process, in which the $A_m^{(\pm)}$ act as jump operators. This process represents ρ as an ensemble of vectors which are, in general, superpositions of different occupation number eigenstates, and fluctuations of $g(t)$ within this ensemble are much smaller. This observation is yet another important result of this paper.

Due to computational constraints we have simulated the case of $N = 10$ particles, confined to the 5 lowest levels of the harmonic trap, for different values of the temperature parameter $q = \Gamma_+/\Gamma_-$. For each value of q, the function $g(t)$ was estimated by averaging over 1500 trajectories.

Figure (1) shows the results of the numerical simulation, compared to the analytic formula (28), for different values of q. Quite remarkably, the analytical result (based in the approximation of $\langle n_0\rangle \gg N - \langle n_0\rangle$) is in very good agreement with the numerical results even for N equal just 10 particles (for larger number N the agreement should be even better). As expected, the agreement is much better for low temperatures than for high temperatures, where the numerical simulation indicates a faster decay of $g(t)$ than that predicted by the low-temperature approximation (28).

VI. CONCLUSIONS

In this paper we have studied quantum dynamics of a Bose gas in a trap undergoing sideband laser cooling in the Lamb-Dicke limit. The master equation describing the dynamics of the system can be regarded as a paradigm equation for collective cooling dynamics. One of the difficult problems associated with such dynamics consists in calculating time dependent correlation functions, such as for instance those that describe temporal phase fluctuations of the Bose condensate. In this paper we have presented a solution to this problem. Our main results should be regarded from the methodological point of view:

- We have formulated an analytic method to describe temporal correlations based on an expansion of the master equation valid at low temperatures, when a large number N_0 of particles are in the condensate. The expansion parameter in our approach is $1/N_0$. The expansion can be, and already has been applied for other models of the Bose gas dynamics, that describe more realistic physical situations (see Refs. [27]).
- We have formulated a numerical method to calculate the time correlations which modifies the jump

processes involved in the master equation in such a way that the corresponding Quantum Monte Carlo simulations are much more stable and require averaging over much less number of quantum trajectories to achieve good accuracy. The method proposed is general, and can be applied not only for the present model, but also for more realistic related models.

- Analytic and numerical results agree very well even at the border of the validity of the analytic theory ($N_0 \simeq 10$).

- Collective laser cooling leads at low temperature to phase diffusion in a Mexican hat potential. This result is also general, and holds for other more realistic models of the collective laser cooling.

The above listed methodological results provide the main value of the present paper. It is, nevertheless, interesting to speculate whether the consider model is purely academic, or whether it can be realized experimentally. We shall argue now that it can be realized for small number ($N = 10 - 50$)) of particles.

Let us discuss step by step the most relevant approximations used in this paper.

- *1 Dimension.* This approximation was done mainly for technical reasons. The model can be easily generalized to describe condensation in a three–dimensional trap. In that case, additional precautions should be taken into account to avoid the reabsorbtion problem. In particular, if the width of the excited states is smaller than $\hbar\omega$, we were thus working not only in the LDL limit, but also in the *festina lente* regime [19]. Extending the dynamics to three dimensions in an asymmetric trap will in this regime not introduce any kind of dangerous reabsorbtion problems. In fact, the dynamics will consist of 3 independent dynamics corresponding to the cooling in the x, y, and z direction.

- *Absence of elastic collisions.* As we have mentioned, this requires that the atomic density should be sufficiently small, or alternatively that the scattering length should be modified to very low values. The first possibility is not interesting, because we require also that the LDL conditions are fulfilled, which means that a should be of the order of at most 0.1μm. The condition to be fulfilled is $4\pi N\hbar^2 a_{sc}/mV < \hbar\omega$, with the effective condensate volume $V = (2\pi)^{3/2}a^3$. For the parameter considered this gives gives $N\zeta < 20$, where ζ is the modification factor of the scattering length.

- *Absence of two body inelastic collisions* This problem has a simple remedy. The cooling and condensation should take place in a dipole trap, and occur in the *electronic ground state* of the atoms. Two body inelastic processes are then completely suppressed.

- *Absence of three body inelastic processes* Three body losses can be typically neglected provided the density is less than $10^{15}/\mathrm{cm}^3$. For $a = 0.1\mu$m that requires $N < (2\pi)^{3/2} \simeq 15$. If the three body losses are modified in a corresponding way as elastic collisions (which seems to be the case for ^{85}Rb [32]), then the corresponding condition is much less restrictive $N\zeta^3 < 15$.

- *Absence of photoassociation losses* This can be reduced by using red detuned laser tuned in between the molecular resonances [30]. Even is such a case, photoassociation losses become relevant when the density reaches the limit $10^{15}/\mathrm{cm}^3$, i.e. in our case for $N < (2\pi)^{3/2} \simeq 15$. It seems likely, that this estimate can be improved significantly when the laser is tuned below the minimum of the molecular transition. While it can hardly be though of for the direct transition, we should stress that laser transition considered in this paper can be equally well regarded as a Raman transition. In such case, tuning of the stimulated two-photon transition below the minimum of the molecular resonance is possible.

Summarizing, we see that even in the worst case (no modifications of the scattering length, no special precautions regarding photoassociation losses) our model should be valid for $N \simeq 10 - 15$ particles. Additional precautions can allow to extent the validity of the model to significantly larger values of N.

VII. ACKNOWLEDGMENTS

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APPENDIX A: ADIABATIC ELIMINATION OF THE NON–CONDENSED STATES

In this Appendix we present in detail the calculations which allow us to transform Eq. (19) into Eq. (20). First, we show that our problem can be reduced to a two–level system formed by the level 0 and 1 of the trap. Then, we adiabatically eliminate the level 1.

1. Two–level system

First, we shall analyze the dynamics of P_1. From (16) and (17) one obtains that

$$\begin{aligned}\dot{P}_1 &= \frac{\Gamma_-}{2}\left[\frac{2}{N}\nabla_0^2 + (q-1)\hat{l}_0\right]P_1^2 \\ &+\frac{\Gamma_-}{2}\left[2(q-1)N|x_0|^2 + (2q+1)\hat{l}_0 - 2\right]P_1 \\ &+\Gamma_- qN|x_0|^2 P_0 \\ &+2\Gamma_-\left[(1-q)P_{1,2}^{1,1} - qP_1 + P_2\right], \end{aligned} \tag{A1}$$

where we have used the notation:

$$\nabla_0^2 = \frac{\partial^2}{\partial x_0 \partial x_0^*}, \tag{A2}$$

$$\hat{l}_0 = \frac{\partial}{\partial x_0}x_0 + \frac{\partial}{\partial x_0^*}x_0^*, \tag{A3}$$

$$P_{1,2}^{j,k} = \int d^2z_1 d^2z_2 \ldots |z_1|^{2j}|z_2|^{2k}P, \tag{A4}$$

and $P_1^2 = P_{1,2}^{2,0}$, $P_2 = P_{1,2}^{0,1}$. Let us analyze in detail the last line in the RHS of Eq. (A1), which comes from the contributions of $\hat{S}_1$ in Eq. (16), i.e. the contributions given by the transitions $1 \leftrightarrow 2$. For a temperature sufficiently below T_c, and sufficiently large N, the atoms in the non–condensed states of the trap form a so–called Maxwell–Demon (MD) ensemble [39], i.e. an ensemble which exchanges particles with a reservoir provided by the condensate without exchanging the energy. Therefore, the excited states can be considered as: (i) independent of the population of the ground state; (ii) decorrelated among each other. Due to these properties:

$$P_{1,2}^{j,k} \simeq \langle n_1^j\rangle_{GC}\langle n_2^k\rangle_{GC}, \tag{A5}$$

where the subindex GC means that the averages are calculated in the grand canonical ensemble. It is well known that these averages have the simple form:

$$\langle n_j\rangle_{GC} = \frac{q^j}{1-q^j} \tag{A6}$$

Therefore, the last line in the RHS side of Eq. (A1) becomes:

$$2\Gamma_-\left[(1-q)\langle n_1\rangle_{GC}\langle n_2\rangle_{GC} - q\langle n_1\rangle_{GC} + \langle n_2\rangle_{GC}\right] = 0. \tag{A7}$$

Therefore the contribution of $\hat{S}_1$ for $\dot{P}_1$ cancels out. In general, the contribution of $\hat{S}_1$ for $\dot{P}_1^n$ is not exactly zero, but it is always a constant, following the MD arguments. Therefore our system reduces to a two–level system, in which only the levels 0 and 1 must be considered.

2. Adiabatic elimination of the level 1

Having reduced the system into just two levels, 0 and 1, we shall adiabatically eliminate the level 1. This can be achieved because, as observed from Eqs. (A1) and (19), P_1 decays on a time scale of the order $\mathcal{O}(1/N)$, whereas P_0 decays in a time scale of the order $\mathcal{O}(1)$. We are interested in contributions up to the order $\mathcal{O}(1/N)$ in $\dot{P}_0$, and hence in contributions up to the order $\mathcal{O}(1/N)$ in the stationary value of P_1. From Eq. (A1) it is clear that the contributions of the order $\mathcal{O}(1)$ in $\dot{P}_1$ lead to terms of the order $\mathcal{O}(1/N)$ in the stationary value of P_1. Therefore we are interested in $\dot{P}_1$ up to order $\mathcal{O}(1)$, and therefore in P_1^2 up to order $\mathcal{O}(1)$. This means that we need to calculate $\dot{P}_1^2$ just up to the order $\mathcal{O}(N)$:

$$\begin{aligned}\dot{P}_1^2 &= \frac{\Gamma_-}{2}\left[-4(1-q)N|x_0|^2\right]P_1^2 \\ &+\frac{\Gamma_-}{2}\left[8qN|x_0|^2\right]P_1. \end{aligned} \tag{A8}$$

For times $t \gg [2\Gamma_-(1-q)N|x_0|^2]^{-1}$, the stationary values:

$$P_1^2 = 2\frac{q}{q+1}P_1. \tag{A9}$$

is obtained. Therefore Eq. (A1) reduces to the form (up to order $\mathcal{O}(1)$):

$$\begin{aligned}\dot{P}_1 &= \frac{\Gamma_-}{2}\left[-2(1-q)N|x_0|^2 + \hat{l}_0 - 2\right]P_1 \\ &+\frac{\Gamma_-}{2}\left[2qN|x_0|^2\right]P_0. \end{aligned} \tag{A10}$$

And for $t \gg [\Gamma_-(1-q)N|x_0|^2]^{-1}$,

$$P_1 \simeq \left[\frac{q}{1-q} + \frac{q}{2N(1-q)^2}\hat{l}_o\frac{1}{|x_0|^2}\right]P_0. \tag{A11}$$

Substituting this value in Eq. (19), one obtains Eq. (20).

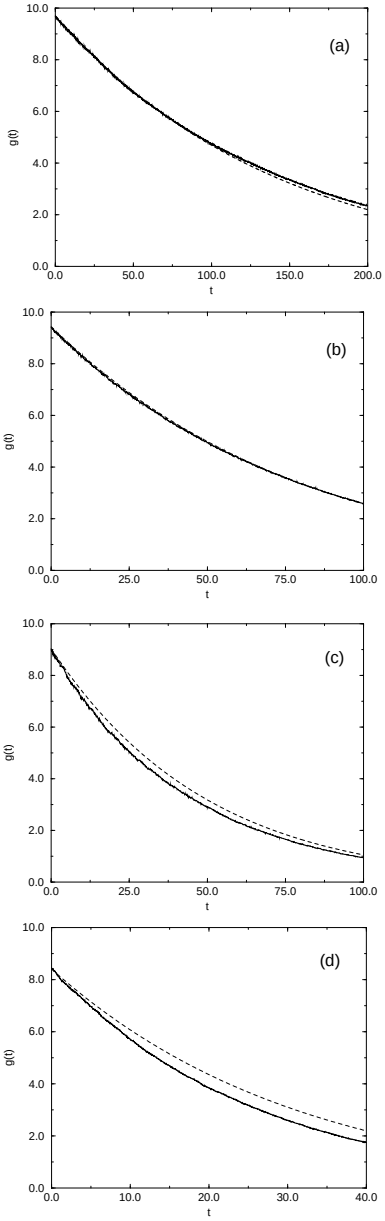

FIG. 1. Comparison between the analytic approximation described by Eq. (28) (dashed line), and the results of the numerical simulations (solid line), for different values of $q = \Gamma_+/\Gamma_-$. In the simulations we have chosen the total particle number $N = 10$. The time scale is in units of Γ_-^{-1}. The numerical results have been obtained by averaging over 1500 trajectories for each value of q.